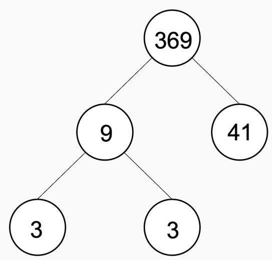
Grade 7/8 Math Circles
February 13/14/15/16, 2023
Prime Time - Solutions

Exercise Solutions

Exercise 1

What is the prime factorization of 369? Use a factor tree in your solution.

Exercise 1 Solution



$\therefore$ the prime factorization of 369 is $3 \times 3 \times 41$.

New Symbol Alert! The $\therefore$ symbol is shorthand for “therefore”. It is often used in mathematical statements.

Exercise 2

Without creating a factorization tree, write the factorization of 56 as a product of powers of its prime factors.

Exercise 2 Solution

$$\begin{aligned}56 &= 8 \times 7\\ &= 2 \times 4 \times 7\\ &= 2 \times 2 \times 2 \times 7\\ &= 2^3 \times 7\end{aligned}$$

Exercise 3

Is 483 divisible by 3? Is it divisible by 11?

Exercise 3 Solution

Using the divisibilty rule for 3:

$4 + 8 + 3 = 15$. Since 15 is divisible by 3 ($15 \div 3 = 5R0$) 483 is divisible by 3.

New Symbol Alert!: the uppercase R in a division statement is shorthand for "remainder".

Using the divisibility rule for 11:

Our alternating sum is $4 - 8 + 3 = -1$. Since -1 is **not** divisible by 11, 483 is not divisible by 11.

Exercise 4

Write out sums that satisfy Goldbach's conjecture for all the even composite numbers up to and including 20. Using the same prime number twice is allowed!

Exercise 4 Solution

$4 = 2 + 2$

$6 = 3 + 3$

$8 = 3 + 5$

$10 = 3 + 7$

$12 = 5 + 7$

$14 = 7 + 7$

$16 = 5 + 11$

$18 = 7 + 11$

$20 = 3 + 17$

Exercise 5

Write out 5 prime pairs that you can think of.

Exercise 5 Solution

3, 5 ; 5, 7 ; 11, 13 ; 17, 19 ; 29, 31

Problem Set Solutions

1. State whether each of the following numbers are prime or composite. If composite, determine the amount of unique prime factors:

   (a) 63

   (b) 114

   (c) 47

   (d) 243

   Solution:

   (a) Composite. 63 has two unique prime factors.

   (b) Composite. 114 has three unique prime factors.

   (c) Prime.

   (d) Composite. 243 has one unique prime factor.

2. How many positive factors does 9690 have?

   Solution:

$$\begin{aligned} 9690 &= 10 \times 969 \\ &= 2 \times 5 \times 3 \times 323 \\ &= 2 \times 5 \times 3 \times 17 \times 19 \end{aligned}$$

Using our formula, the number of positive factors of 9690 is:

$(a_0 + 1) \times (a_1 + 1) \times (a_2 + 1) \times (a_3 + 1) \times (a_4 + 1)$
$= (1 + 1) \times (1 + 1) \times (1 + 1) \times (1 + 1) \times (1 + 1)$
$= 2 \times 2 \times 2 \times 2 \times 2$
$= 32$

3. What fraction of integers between 1 and 30, inclusive, is prime?

Solution: Since our range is quite small, we can list all the prime numbers between 1 and 30, inclusive, count them, and write it as a fraction over 30:
2, 3, 4, 7, 11, 13, 17, 19, 23, 29. Thus there are 10 primes between 1 and 30, inclusive.

Since there are 10 primes and 30 numbers in total, $\frac{10}{30} = \frac{1}{3}$ of integers between 1 and 30, inclusive, are prime.

4. The *greatest common divisor* (GCD) of two numbers a, b is the largest number that divides into both a and b. List the GCD for all the pairs of even numbers between 20 and 30.

Solution:

Pair	GCD
20, 22	2
20, 24	4
20, 26	2
20, 28	4
20, 30	10

Pair	GCD
22, 24	2
22, 26	2
22, 28	2
22, 30	2
24, 26	2

Pair	GCD
24, 28	4
24, 30	6
26, 28	2
26, 30	2
28, 30	2

5. What is the difference between the two greatest prime factors of 585?

Solution:

$$\begin{aligned} 585 &= 5 \times 117 \\ &= 5 \times 3 \times 39 \\ &= 5 \times 3 \times 3 \times 13 \end{aligned}$$

The two greatest prime factors of 585 are 5 and 13, so the difference between them is $13 - 5 = 8$.

6. Determine the smallest integer with exactly five unique factors.

Solution: First, recall that factors of numbers come in pairs. Both 3 and 4 are factors of 12, and since $4 \times 3 = 12$, they are a "pair". When we list all of the factors of an integer, we can easily pair them up as well.
Of the factors of 12: 1, 2, 3, 4, 6, 12,
The pairs are: 1, 12 ; 2, 6 ; 3, 4

It seems then that we should have an *even* amount of factors; so what would need to be true in order to have an odd number of unique factors?
A pair of factors would have to consist of the same number! Consider the factors of 9: 1, 3, 3, 9. The factor "3" appears twice, so really there are only three unique factors, not four. We can also notice that the existence of a factor whose "pair" is itself, means that our integer is a *perfect square*! This tells us that in order to find a number with exactly five factors, it must be a perfect square.

The only perfect squares less than 9 are 1 and 4; 1 has one divisor (1) and 4 has three divisors (1, 2 and 4) so they do not satisfy what the question is asking. The next perfect square after 9 is 16; we can investigate its factors to see if there are five unique integers:

1, 2, 4, 4, 8, 16 $\rightarrow 1, 2, 4, 8, 16$

We can see that 16 has exactly five unique factors, and is the first perfect square to satisfy this property, thus it must be the smallest integer with exactly five unique factors.

7. The seven-digit number $6,227,\underline{d}32$ is divisible by 11. What is the digit d?

Solution: For $6,227,\underline{d}32$ to be divisible by 11, the alternating sum of its digits must be divisible by 11:
$6-2+2-7+d-3+2=d-2$
So for some non-negative integer k, $d-2=11k$, where $11k$ is a multiple of 11.

W know d musy be between 0 and 9, inclusive, and so the largest that $11k$ can be is $9-2=7$. But 7 is less than 11, so it cannot be that $d-2$ is equal to 11 or a larger multiple of 11. It may seem that we are stuck and cannot continue further, but there is one more multiple of 11 that we can consider: 0! Indeed, $0 \div 11 = 0$ (just as every other number also divides into 0).

This tells us that we need $d-2=0 \rightarrow d=2$, since $2-2=0$. So, in order for $6,227,\underline{d}32$ to be divisible by 11, the digit d must be 2.

8. What are the possible k values for the four-digit number $561k$ if its prime factorization must include:

(a) 3

(b) 2

(c) At least two powers of 3

(d) 3 **and** 7

Solution:

(a) If the prime factorization of $561k$ includes 3, then it must be divisible by 3. Using our divisibility rule, we know that the sum of the digits of $561k$ must be divisible by 3 as well:

$5+6+1+k=12+k$

Since 12 is a multiple of 3, k must be as well since only a multiple of 3 added to 12 will result in another multiple of 3. Thus k has four possible values: 0, 3, 6, or 9.

(b) According to our divisibility rules, for $561k$ to be divisible by 2, k must be an even number. Thus k has five possible values: 0, 2, 4, 6 or 8.

(c) If the prime factorization of $561j$ included at least two powers of 3, then it must have a factor of 9 since $3^2 = 9$. Following the divisibility rule for 9, we must again look at the sum of the digits of $561k$:

$6+6+1+k = 12+k$. The closes multiples of 9 are 18 and 27; we can check if there exists a k value that satisfies this:

$12 + 6 = 18$, so k can be 6, but $27 = 12 + 15$; since k must be between 0 and 9, the only possibility is that $k = 6$.

(d) For $561k$ to be divisible by both 3 *and* 7, it must satisfy the divisibility rules for both numbers. From part a), we already have possible k values limited to 0, 3, 6 or 9. We can thus look at each of 5610, 5613, 5616 and 5619 to determine which ones (if any) satisfy the divisibility rule for 7:

	$k = 0$	$k = 3$	$k = 6$	$k = 9$
Use rule	$561-2(0) = 561$	$561-2(3) = 555$	$561-2(6) = 549$	$561-2(9) = 543$
Use rule	$56-2(1) = 54$	$55-2(5) = 45$	$54-2(9) = 36$	$54-2(3) = 48$
Recall multiples of 7	54 is *not* a multiple of 7	45 is *not* a multiple of 7	36 is *not* a multiple of 7	48 is *not* a multiple of 7

Since none of 0, 3, 6 or 9 satisfy the divisibility rule for 3 *and* 7, there is no value of k such that $561k$ is divisible by 7.

9. The three digit number $3a8$ is added to 243 and gives $6b1$. If $6b1$ is divisible by 9, find the value of $a \times b$.

Solution: In order to determine $a \times b$, we need to figure out the values of a and b.
We know that:

$$\begin{array}{r} 3\,a\,8 \\ +\,2\,4\,3 \\ \hline 6\,b\,1 \end{array}$$

Evidently, $2 + 3 \neq 6$. Thus to get 6 in the hundredths place, we know that $a + 5$ has to be greater than or equal to 10. *(it is* $a+5$ *because* $8+3=11$ *and so we carry the* 1*).* This tells us that we must carry the 1 from the tens place to the hundredths place, which leaves us with $a+5-10=a-5$. So $b=a-5$.

Now we can consider our condition that $6b1$ is divisible by 9. Our divisibility rule tells us that $6+b+1$ must be divisible by $9 \rightarrow 7+b$ is divisible by 9. Looking at multiples of 9: 9, 18, 27, ...
$7+2=9 \rightarrow b$ is 2
$7+11=18 \rightarrow b$ is 11, but b must be less than or equal to 10.
$\therefore b=2$ is our only possibility which means that $a=7$ since $2=7-5$. Thus we have:

$$a \times b = 7 \times 2 = 14.$$

10. The product of three different positive integers is 168. What is the largest possible sum of these three integers? *(Note: the integers must be greater than 1)*

Solution: First we find the prime factorization of 168:

$$\begin{aligned} 168 &= 2 \times 2 \times 2 \times 3 \times 7 \\ &= 2^3 \times 3 \times 7 \end{aligned}$$

$$\begin{aligned} \therefore \text{Total \# factors} &= (3+1) \times (1+1) \times (1+1) \\ &= 4 \times 2 \times 2 \\ &= 16 \end{aligned}$$

The following are the 16 factors of 168:

1, 2, 3, 4, 6, 7, 8, 12, 14, 21, 24, 28, 42, 56, 84, 168.

Since we must have three numbers none of which are 1, we can eliminate the pair (1, 168) and the number 84 (since we would have $84 \times 2 \times 1 = 168$). We can also exclude having 56 in our three numbers since $56 \times 3 \times 1 = 168$ and we can't break down 3 since it is prime. We can now go through each of our factors starting from the next largest ones, and consider the different combinations of numbers and their respective sums:

next largest is 42, and then we have 28, and then 24, and then 21, and then 14, and then 12, and then we're sort of back where we started sooo that is it.

Largest factor	Combinations with largest factor	Sum
42	$42 \times 4 \to 42 \times 2 \times 2$	N/A*
28	$28 \times 6 \to 28 \times 3 \times 2$	33
24	$24 \times 7 \to 24 \times 7 \times 1$	N/A*
21	$21 \times 8 \to 21 \times 4 \times 2$	27
14	$14 \times 12 \to 14 \times 6 \times 2$	22
	$14 \times 12 \to 14 \times 4 \times 3$	20

*N/A placed if the combination is not valid due to having a duplicate number or the number 1.

Evidently, the largest possible sum of three unique integers whose product is 168 is $28 + 2 + 3 = 33$.

11. Mr. Math has a box of protractors with a volume of 858cm^3. What are the possible dimensions of the box?
(Recall: Volume = length × width × height)

Solution: As with Problem 10, we first look at the prime factorization of our number, 858:

$$858 = 2 \times 3 \times 11 \times 13$$

$$\begin{aligned} \therefore \text{Total \# factors} &= (1+1) \times (1+1) \times (1+1) \times (1+1) \\ &= 2^4 \\ &= 16 \end{aligned}$$

We can now break down this question into different cases; in particular, we will look at dimensions that include 1cm and dimensions that do not include 1cm as a side length:

1cm is the length of at least one dimension

If 1 is a dimension, the other two can just be factor pairs of 858. Factors of 858 are:
1, 2, 3, 6, 11, 13, 22, 26, 33, 39, 66, 78, 143, 286, 429, 858.

So the following dimensions are possible, for a total of 8 possibilities:

$1 \times 1 \times 858$	$1 \times 11 \times 78$
$1 \times 2 \times 429$	$1 \times 13 \times 66$
$1 \times 3 \times 286$	$1 \times 22 \times 39$
$1 \times 6 \times 143$	$1 \times 26 \times 33$

1cm is not the length of any of the dimensions

If 1 is not a dimension, we need to split up our pairs:

$2 \times 429 \rightarrow$	$2 \times 3 \times 143$	$3 \times 286 \rightarrow$	$3 \times 2 \times 143^*$
	$2 \times 33 \times 13$		$3 \times 22 \times 13$
	$2 \times 39 \times 11$		$3 \times 26 \times 11$
$6 \times 143 \rightarrow$	$2 \times 3 \times 143^*$	$11 \times 78 \rightarrow$	$11 \times 2 \times 39^*$
	$6 \times 11 \times 13$		$11 \times 6 \times 13^*$
			$11 \times 3 \times 26^*$
$13 \times 66 \rightarrow$	$13 \times 6 \times 11^*$	$22 \times 39 \rightarrow$	$2 \times 11 \times 39^*$
	$13 \times 22 \times 3^*$		$22 \times 3 \times 13^*$
	$13 \times 33 \times 2^*$		
$26 \times 33 \rightarrow$	$2 \times 13 \times 33^*$		
	$26 \times 3 \times 11^*$		

* *repeated combinations*

We have 6 unique triples that do not include 1 (not taking order into account) and 8 triples that include 1. Thus in total, we have 14 possible dimensions as listed above.